

Among problems involving wave motion from moving sources more attention is being paid to the radiation of internal waves when the moving sources are accelerated [1, 2]. In the present paper we consider the total energy of radiation and its spectral distribution from mass sources undergoing periodic motion. The methods and basic notation are as in [3, 4].*

1. Wave Radiation for Motion of the Source along a Helix. In a uniformly stratified ideal incompressible fluid, the general expression for the energy loss of a mass source per unit time is given by [3, 4]

$$W = \int d^3r p(\mathbf{r}, t) m(\mathbf{r}, t),$$

If we make use of the proportionality (in the linear description) between the pressure perturbation p and the mass source m inducing it, we can rewrite w as a quadratic form in the Fourier transform $m(\mathbf{k}, \omega)$ of the mass source:

$$W = \frac{i}{(2\pi)^5} \int d^3k d\omega d\sigma \omega (N^2 - \omega^2) G^{\text{ret}}(\mathbf{k}, \omega) e^{-i(\omega + \sigma)t} m(\mathbf{k}, \omega) m(-\mathbf{k}, \sigma). \quad (1.1)$$

Where $G^{\text{ret}}(\mathbf{k}, \omega)$ is the Fourier transform of the scalar retarded Green's function for the internal wave equation. We consider the case where the Brunt-Väisälä frequency N is constant. Then in the Boussinesque approximation we have

$$G^{\text{ret}}(\mathbf{k}, \omega) = [(\omega + i\epsilon)^2 k^2 - N^2 k_h^2]^{-1}, \quad (1.2)$$

where ω is the frequency and ϵ is an infinitesimal positive constant put in as usual to avoid the singularities on the real axis in correspondence with the principle of causality. \mathbf{k} is the wavevector and k_h is its horizontal component.

For a point source of constant intensity m_0 uniformly incident along a helix

$$\begin{aligned} m(\mathbf{r}, t) &= m_0 \delta(\mathbf{r} - \mathbf{R}(t)), \\ \mathbf{R}(t) &= (R_0 \sin \omega_0 t, R_0 \cos \omega_0 t, v_0 t), \end{aligned} \quad (1.3)$$

With the help of the well-known expansion

$$e^{-i\xi \sin \alpha} = \sum_{n=-\infty}^{+\infty} J_n(\xi) e^{-in\alpha}, \quad (1.4)$$

where the $J_n(\xi)$ are the Bessel functions, we can write $m(\mathbf{r}, t)$ in Fourier series form

$$m(\mathbf{k}, \omega) = 2\pi m_0 \sum_{n=-\infty}^{+\infty} J_n(k_h R_0) e^{-in\varphi} \delta(\omega - n\omega_0 - k_z v_0). \quad (1.5)$$

where φ is the angular coordinate of vector \mathbf{k}_h in the horizontal plane.

Substituting series (1.5) into (1.1) and integrating with respect to φ and the frequency σ we get

$$W = \frac{im_0^2}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \int_0^\infty dk_h k_h J_n^2(k_h R_0) \int d\omega dk_z \omega (N^2 - \omega^2) G^{\text{ret}}(\mathbf{k}, \omega) \delta(\omega - k_z v_0 - n\omega_0). \quad (1.6)$$

and so the expression for the energy loss in the present case is independent of time.

*Another method of calculating the energy loss of moving sources based on the asymptotic values of the wave amplitudes is discussed in the monograph "Waves in Fluids" by J. Lighthill, Mir, Moscow (1981).

From the fact that W is real it follows that only the imaginary part of the Fourier-transformed retarded Green's function survives. From (1.2) we have

$$\text{Im } G^{\text{ret}}(\mathbf{k}, \omega) = -\pi \text{sgn } \omega \delta(\omega^2 \mathbf{k}^2 - N^2 \mathbf{k}_h^2). \quad (1.7)$$

Hence the energy loss of a point source moving along a helix can be rewritten in the form

$$W = \frac{m_0^2}{4\pi} \sum_{n=-\infty}^{+\infty} \int d\omega dk_z \int_0^\infty dk_h k_h J_n^2(k_h R_0) |\omega| (N^2 - \omega^2) \delta(\omega^2 \mathbf{k}^2 - N^2 \mathbf{k}_h^2) \delta(\omega - k_z v_0 - n\omega_0). \quad (1.8)$$

Because of the two delta-functions the number of integrations can be reduced to one; further simplification depends on whether the vertical component of the velocity of motion v_0 is equal to zero or not.

We consider the more interesting case where the source moves uniformly around a horizontal circle of radius R_0 so that $v_0 = 0$.* Integrating (1.8) with respect to k_z and ω we obtain

$$W = \frac{m_0^2}{2\pi} \sum_{n=1}^{\lfloor N/\omega_0 \rfloor} \sqrt{N^2 - n^2 \omega_0^2} \int_0^\infty dk_h J_n^2(k_h R_0). \quad (1.9)$$

We note that out of the entire harmonic series there remains a finite number of terms and an upper bound on the number of terms is given by the ratio of the angular velocity of the source around the circle to the Brunt-Väisälä frequency N ($n \leq N/\omega_0$). If the angular velocity of revolution is larger than N , then in general there will be no radiation (this is typical of harmonic excitation of internal waves; see Secs. 2 and 3 in [6]). Since the ratio ω_0/N can be thought of as a rotational Froude number V/NR_0 , the radiation condition $\omega_0 < N$ is the assertion that the rotational Froude number be less than unity.

We turn now to the integral with respect to the wavenumber k_h . This integral will be logarithmically divergent at large k_h because of the large contribution of very short wavelengths in the radiation of a point source. For more realistic sources of finite spatial extent, the contribution of wavenumbers exceeding $1/r_0$, where r_0 is a characteristic spatial dimension of the source, will vanish. It can be shown that for an arbitrary nonlocal source of the form $m(\mathbf{r}, t) = m_0 f(\mathbf{r} - \mathbf{R}(t))$ in the general expression (1.6) for the energy loss averaged over a period $2\pi/\omega_0$, the extra factor $|f(\mathbf{k})|^2$ appears. This factor falls off with large wavenumbers rapidly enough to ensure that the entire integrand is such that the integral converges.

The weak logarithmic nature of the divergence of the integral (1.9) means that the final results will be only weakly dependent on the details of the source. Therefore one can estimate the integral by cutting it off for wavenumbers exceeding the upper limit $k_h \sim 1/r_0$. Then the asymptotic value of the integral for $R_0/r_0 \gg 1$ leads to the result

$$W \approx \frac{m_0^2}{2\pi^2 R_0} \ln \frac{R_0}{r_0} \sum_{n=1}^{\lfloor N/\omega_0 \rfloor} \sqrt{N^2 - n^2 \omega_0^2}. \quad (1.10)$$

We compare this with another limiting case where the point mass source moves uniformly in the vertical direction. Putting $R_0 = 0$, $\omega_0 = 0$ in (1.6) and integrating with respect to the wavenumber we find (see also [7])

$$W = \frac{m_0^2}{8\pi r_0} \int_{-N}^{+N} d\omega |\omega| = \frac{m_0^2 N^2}{8\pi r_0}. \quad (1.11)$$

From (1.11) and (1.10) it is clear that when $\omega_0 \leq N$, $v_0 \sim V = \omega_0 R_0$ the energy loss for motion of the source around a circle has the same order of magnitude as the energy loss in radiation of internal waves for uniform rectilinear motion.

If the wave source is a point dipole (doublet) with the dipole moment vector directed along the velocity vector ($\mathbf{d} = \mu_0 \mathbf{V}$), then

*In electrodynamics, the radiation of electromagnetic waves by a charge for this geometry is known as synchrotron radiation [5].

$$m(\mathbf{r}, t) = -\mathbf{dV}\delta(\mathbf{r} - \mathbf{R}(t)) = -\mu_0 \mathbf{V}\mathbf{V}\delta(\mathbf{r} - \mathbf{R}(t)) = \mu_0 \frac{\partial}{\partial t} \delta(\mathbf{r} - \mathbf{R}(t))$$

and obviously an extra factor of $-i\omega$ appears in the expression for $m(\mathbf{k}, \omega)$ in (1.5) and in the integrand of (1.6) we have the corresponding additional factor of ω^2 .

For a doublet moving uniformly around a horizontal circle we find from (1.6) in the same way as above

$$W \approx \frac{\mu_0^2}{2\pi^2 R_0} \ln \frac{R_0}{r_0} \sum_{n=1}^{[N/\omega_0]} n^2 \omega_0^2 \sqrt{N^2 - n^2 \omega_0^2}.$$

For comparison it follows from (1.6) that for uniform vertical motion of the doublet we have (see also [7])

$$W = \mu_0^2 N^4 / (16\pi v_0). \quad (1.12)$$

From the well-known modeling of a sphere of radius r_0 by a doublet with $\mu_0 = 2\pi r_0^3$ in a uniform fluid, the last result can be used to estimate the radiation of internal waves by a sphere in a weakly stratified fluid (for $Nr_0 \ll V$, $Nr_0 \ll v_0$).

2. Radiation for Constant Intensity Vibration of the Mass Source. We consider the energy loss of a mass source undergoing periodic motion of a different form. Let us consider a uniform motion of the source in one direction superimposed on vibrational motion of frequency ω_0 in another direction \mathbf{a} . Then using (1.4) for a point source we have (see (1.3) and (1.5)):

$$m(\mathbf{r}, t) = m_0 \delta(\mathbf{r} - \mathbf{R}(t)), \quad \mathbf{R}(t) = \mathbf{v}_0 t + \mathbf{a} \sin \omega_0 t,$$

$$m(\mathbf{k}, \omega) = 2\pi m_0 \sum_{n=-\infty}^{+\infty} J_n(\mathbf{k}\mathbf{a}) \delta(\omega - n\omega_0 - \mathbf{k}\mathbf{v}_0).$$

We substitute the latter expansion in the general formula for the energy loss (1.1) and average over a vibrational period $2\pi/\omega_0$, integrate with respect to frequency σ , and substitute (1.7). We then obtain

$$\langle W \rangle = \frac{m_0^2}{8\pi^2} \sum_{n=-\infty}^{+\infty} \int d^3k d\omega |\omega| (N^2 - \omega^2) J_n^2(\mathbf{k}\mathbf{a}) \delta(\omega^2 \mathbf{k}^2 - N^2 \mathbf{k}_h^2) \delta(\omega - \mathbf{k}\mathbf{v}_0 - n\omega_0), \quad (2.1)$$

which is analogous to (1.8).

In the special case of vertical vibrations ($\mathbf{a} = (0, 0, a)$, $\mathbf{v}_0 = 0$) we integrate with respect to the horizontal component of the wavevector and this formula simplifies to

$$\langle W \rangle = \frac{m_0^2}{8\pi} \sum_{n=-\infty}^{+\infty} \int dk_z \int_{-N}^N d\omega |\omega| \delta(\omega - n\omega_0) J_n^2(k_z a).$$

Replacing the upper limit of integration by $|k_z| \sim 1/r_0$ to eliminate the divergence as done in Sec. 1 above, we perform the integral with respect to ω and find

$$\langle W \rangle \approx \frac{m_0^2 \ln a/r_0}{4\pi a} \sum_{n=1}^{[N/\omega_0]} n\omega_0. \quad (2.2)$$

From comparison of this result with (1.10) we see the analogy for the types of periodic motion of point sources considered here. This analogy also exists for other types of sources.

3. Radiation for Motion of the Source about an Elliptical Trajectory. By analogous methods we can treat the nonuniform motion of the source about an elliptical trajectory with constant angular velocity ω_0 . For a point source moving in the horizontal plane along the curve $x^2/a^2 + y^2/b^2 = 1$, we have

$$m(\mathbf{r}, t) = m_0 \delta(\mathbf{r} - \mathbf{R}(t)), \quad \mathbf{R}(t) = (a \sin \omega_0 t, b \cos \omega_0 t, 0),$$

and with the help of (1.4) we find

$$m(\mathbf{k}, \omega) = 2\pi m_0 \sum_{n=-\infty}^{+\infty} J_n(\kappa) e^{-in\varphi} \delta(\omega - n\omega_0), \quad (3.1)$$

$$\kappa = |\boldsymbol{\kappa}|, \quad \boldsymbol{\kappa} = (k_x a, k_y b, 0), \quad \cos \varphi = ak_x/\kappa.$$

After substitution of this expansion in the quadratic form (1.1) and carrying out similar steps to those above (averaging over the period of motion $2\pi/\omega_0$, replacement of the Green's function by its imaginary part (1.7), and integrating over frequency and the vertical component of the wavevector) we obtain the final result

$$\langle W \rangle = \frac{m_0^2}{4\pi^2} \sum_{n=1}^{[N/\omega_0]} \sqrt{N^2 - n^2\omega_0^2} \int d^3k_h \frac{J_n^2(\kappa)}{k_h}. \quad (3.2)$$

As expected, in the special case of circular motion ($a=b$) this result reduces to (1.9) except that in (3.2) the energy loss has been averaged over the period of motion.

In the other limiting case $b=0$ (or $a=0$) formula (3.2) describes radiation for a source vibrating along a horizontal axis. It can be seen that the spectrum of the radiation is practically independent of the details of the motion. The entire difference is in the value of the cut-off integral with respect to the wavenumber which is asymptotically independent of the number n ($J_n^2(\xi) \approx \frac{2}{\pi\xi} \cos^2 \xi$ for $\xi \gg 1$) when the dimension of the source r_0 is small in comparison with the scale of motion.

4. Appendix. Another Method of Calculating the Radiation. Above we calculated the energy loss of moving sources per unit time. From conservation of energy, for a uniformly moving source in an ideal fluid, the energy loss is equal to the radiation energy of internal waves [3, 4], which is defined as the flux of energy through a surface surrounding the source. The same is true of periodic motion when we average over time. It is demonstrated below by actual calculation the equivalence of the two methods for the case of uniform vertical motion of a point source when the energy loss is finite (see (1.11), (1.12)).

The energy flux density vector $\mathbf{S} = p(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t)$, integrated over the time of flight of the source is

$$\int dt \mathbf{S} = \frac{1}{2\pi} \int d\omega p(\mathbf{r}, \omega) \mathbf{v}(\mathbf{r}, -\omega). \quad (4.1)$$

Appearing in this formula are the small perturbations in the pressure and velocity which can be expressed in terms of the mass source $m(\mathbf{r}, \omega)$ creating them and the "potential" $\psi_\omega \equiv \psi(\mathbf{r}, \omega)$ according to the formulas [3]

$$(N^2 \nabla_h^2 - \omega^2 \Delta) \psi_\omega = -m(\mathbf{r}, \omega),$$

$$p(\mathbf{r}, \omega) = -i\omega(N^2 - \omega^2)\psi_\omega, \quad \mathbf{v}(\mathbf{r}, \omega) = (\omega^2 \nabla - N^2 \nabla_h) \psi_\omega. \quad (4.2)$$

Using these relations, the horizontal component of the energy flux vector (4.1) is

$$\int dt S_h = \frac{i}{2\pi} \int d\omega \omega (N^2 - \omega^2)^2 \psi_\omega \nabla_h \psi_{-\omega}. \quad (4.3)$$

For a point source moving in the vertical direction we have

$$m(\mathbf{r}, t) = m_0 \delta(x) \delta(y) \delta(z - v_0 t), \quad m(\mathbf{r}, \omega) = \frac{m_0}{v_0} \delta(x) \delta(y) e^{iz\omega/v_0}$$

and the solution of (4.2) can be written with the help of the Fourier transform of the Green's function $G^{\text{ret}}(\mathbf{r}_h, k_z, \omega)$ as follows:

$$\psi_\omega = -\frac{m_0}{v_0} e^{iz\omega/v_0} G^{\text{ret}}\left(\mathbf{r}_h, \frac{\omega}{v_0}, \omega\right) \equiv -\frac{m_0}{v_0} e^{iz\omega/v_0} G,$$

Then since \mathbf{S} is real we can transform (4.3) to the form

$$\int dt S_h = -\frac{m_0^2}{2\pi v_0^2} \int d\omega \omega (N^2 - \omega^2)^2 (\text{Im} G \nabla_h \text{Re} G - \text{Re} G \nabla_h \text{Im} G). \quad (4.4)$$

In a fluid where the Brunt-Väisälä frequency is constant, the function $G^{\text{ret}}(\mathbf{r}_h, k_z, \omega)$ satisfies the equation

$$\{N^2 \nabla_h^2 - (\omega - i\varepsilon)^2 (\nabla_h^2 - k_z^2)\} G^{\text{ret}}(\mathbf{r}_h, k_z, \omega) = \delta(\mathbf{r}_h) \quad (4.5)$$

and can be written in terms of cylindrical functions

$$G^{\text{ret}}(\mathbf{r}_h, k_z, \omega) = \frac{\theta(N^2 - \omega^2)}{4(N^2 - \omega^2)} \{N_0(\rho) - i \operatorname{sgn} \omega J_0(\rho)\} + \frac{\theta(\omega^2 - N^2)}{2\pi(\omega^2 - N^2)} K_0(\rho), \quad (4.6)$$

$$\rho \equiv r_h \frac{|\omega k_z|}{\sqrt{|\omega^2 - N^2|}}.$$

Substituting this expression into (4.4) and using the well-known relation for cylindrical functions

$$J_0(\rho) \frac{\partial}{\partial \rho} N_0(\rho) - N_0(\rho) \frac{\partial}{\partial \rho} J_0(\rho) = \frac{2}{\pi \rho} \quad (4.7)$$

we can simplify (4.4) to the form

$$\int dt \mathbf{S}_h = \frac{m_0^2}{16\pi^2 \varepsilon_0^2} \frac{r_h}{r_h^2} \int_{-N}^N d\omega |\omega| = \left(\frac{m_0 N}{4\pi \varepsilon_0 r_h} \right)^2 r_h. \quad (4.8)$$

We note that in the simplified result only $\operatorname{Im} G^{\text{ret}}(\mathbf{r}_h, k_z, \omega)|_{r_h=0} = -1/4(N^2 - \omega^2)^{-1} \operatorname{sgn} \omega \theta(N^2 - \omega^2)$, is actually necessary rather than the entire expression (4.6). It follows from (4.5) that

$$[(N^2 - \omega^2) \nabla_h^2 + \omega^2 k_z^2] \operatorname{Re} G = \delta(\mathbf{r}_h), \quad [(N^2 - \omega^2) \nabla_h^2 + \omega^2 k_z^2] \operatorname{Im} G = 0.$$

Multiplying the first equation by $\operatorname{Im} G$ and the second by $\operatorname{Re} G$ and subtracting we find

$$\nabla_h (\operatorname{Im} G \nabla_h \operatorname{Re} G - \operatorname{Re} G \nabla_h \operatorname{Im} G) = \frac{\operatorname{Im} G}{N^2 - \omega^2} \delta(\mathbf{r}_h)$$

and we will then have

$$\operatorname{Im} G \nabla_h \operatorname{Re} G - \operatorname{Re} G \nabla_h \operatorname{Im} G = \frac{\operatorname{Im} G(\mathbf{r}_h=0, k_z, \omega)}{N^2 - \omega^2} \nabla_h \frac{\ln r_h}{2\pi}. \quad (4.9)$$

Hence $\operatorname{Im} G|_{r_h=0}$ goes in the integrand in (4.4). Finally, all the steps in the derivation are equivalent to those done previously because using (4.6), relation (4.9) reduces to (4.7).

If we now consider the projection of \mathbf{S}_h onto the direction \mathbf{r}_h and integrate (4.8) over the surface of a cylindrical segment of unit height and radius r_h , we obtain the total energy of radiation per unit path length

$$2\pi r_h \int dt \mathbf{S}_h = \frac{m_0^2 N^2}{8\pi \varepsilon_0^2}.$$

and if we multiply this by v_0 we obtain formula (1.11) for the energy loss per unit time.

LITERATURE CITED

1. I. V. Sturova, "Internal waves in an exponentially stratified fluid for an arbitrary motion of source," *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 3 (1980).
2. E. W. Graham and B. B. Graham, "The tank wall effect on internal waves due to a transient vertical force moving at fixed depth in a density-stratified fluid," *J. Fluid Mech.*, 97, No. 1 (1980).
3. V. A. Gorodtsov and É. V. Teodorovich, "Radiation of internal waves for uniform rectilinear motion of local and nonlocal sources," *Izv. Akad. Nauk SSSR*, 16, No. 9 (1980).
4. V. A. Gorodtsov and É. V. Teodorovich, "Planar problem for internal waves produced by moving singular sources," *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 2 (1981).
5. I. M. Ternov, V. V. Mikhailin, and V. R. Khalilov, *Synchrotron Radiation and its Application* [in Russian], Moscow State Univ. (1980).
6. D. E. Mowbray and B. S. H. Rarity, "A theoretical and experimental investigation of the phase configuration of internal waves of small amplitude in a density-stratified liquid," *J. Fluid. Mech.*, 28, No. 1 (1967).

7. V. A. Gorodtsov, "Radiation of internal waves for the vertical motion of a body through a nonuniform fluid," *Inz.-fiz. Zh.*, 39, No. 4 (1980).

PROBABILITY DISTRIBUTIONS OF THE VELOCITY
FLUCTUATIONS IN AXISYMMETRICAL TURBULENT WAKES

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Experimental data are reported on the one-dimensional probability distribution functions and up to the sixth statistical moments of the turbulent velocity fluctuations in hydrodynamic wakes of bluff and streamlined bodies. The data complement similar existing information for various turbulent flows: after a grid [1, 2]; in a two-dimensional wake [3]; in circular [4] and plane [5] jets; in a boundary layer [6]; in a circular pipe [7], etc. The problems of self-similarity of the investigated flow, the influence of the conditions of its evolution on the fluctuation characteristics in the self-similarity zone, and the role of intermittency at the wake boundary are discussed on the basis of the experimental data.

1. Experiments have been carried out in a low-turbulence wind tunnel with the application of a DISA Elektronik hot-wire anemometer system with a linearizer. Either a sphere of diameter $D = 1$ cm or a body of revolution (set up at zero angle of attack) with a midsection diameter $D = 1$ cm and an 8:1 elongation was suspended on wires of diameter 0.05 mm in the tunnel working section, which had a length of 4 m and a cross section of 40×40 cm and was fitted with triangular moldings in the corners to diminish secondary flows. In both cases the Reynolds number $Re = U_\infty D / \nu = 10^4$ (where U_∞ is the freestream velocity and ν is the kinematic viscosity coefficient). Measurements have shown that this value of Re is large enough for the flow in the wake of the sphere to be self-similar with respect to the longitudinal coordinate and, hence, for similarity to hold with respect to the Reynolds number. To obtain similarity with respect to Re and self-similarity in the wake of the elongated body a turbulence generator in the form of a ring of diameter 8 mm and thickness 0.5 mm was set up in the bow region of the body. As a result, the drag forces F_x on the profiled body and the sphere did not differ appreciably, and so the drag coefficients c_x defined by the relation

$$F_x = c_x \rho S U_\infty^2 / 2, \quad S = \pi D^2 / 4,$$

were equal to 0.39 and 0.48 respectively. The small difference in the drag forces fit in quite well with one of the objectives of the experiments, which was to show that the characteristics of a wake in the self-similar region are not determined solely by the drag and free-stream velocity, but depend strongly on the configuration of the body.

Below, we use a cylindrical coordinate system x, r, θ , which is attached to the body with its origin located at the trailing edge of the body and its x axis directed downstream. In addition to the constants U_∞ and D , we also use the following functions of x as typical scales of the velocity and length:

$$U_c(x) = U_\infty \left(\frac{x - x_0}{\sqrt{c_x S}} \right)^{-2/3}, \quad l_c(x) = \sqrt{c_x S} \left(\frac{x - x_0}{\sqrt{c_x S}} \right)^{1/3},$$

which are based on considerations of self-similarity of the flow. Here x_0 is the virtual origin of the wake and in the given experiments is close to zero for both bodies [8].

The probability density function $p(e)$ of the stationary (in the statistical sense) hot-wire signal $e(t)$ was estimated by means of an Intertechnique Histomat-S random-process analyzer. The signal $e(t)$ was related to the longitudinal component of the velocity $u(t)$ in the wake by the linear equation $e = a + ku$, where a and k are constants determined in static calibration of the hot-wire anemometer. The following statistical characteristics were determined in subsequent processing on a general-purpose computer: the probability density function of the velocity fluctuations